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The influence of non-constant diffusivities on solar ponds stability

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Abstract—A solar pond is a basin of water where solar energy is trapped due to an artificially created gradient of salinity that prevents convective motions. The present study intends to clarify the contribution of non-constant diffusion coefficients for the stability of the gradient layer together with the influence of solar radiation absorption, the thermal and molecular diffusivities being assumed to be linear functions of the vertical co-ordinate z. The analysis shows that the consideration of these two effects decreases the margin of stability in comparison with previous studies based on a layer of fluid heated from below with constant diffusivities coefficients and linear profiles for both temperature and salt. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

A solar pond is a basin of water where a salinity gradient is artificially created in order to prevent convective motions induced by solar radiation absorption.

In the gradient layer of a solar pond temperature and salt diffusion proceed with different rates. This difference is responsible for the presence of oscillatory movements in the layer that can be amplified and can degenerate into pure convective motions even in the situation where the layer is gravitically stable, i.e. when density increases towards the bottom.

The marginal states for the beginning of these oscillatory movements were obtained previously by Veronis [1], for a layer of fluid with constant diffusion coefficients and, therefore, constant gradients of temperature and salt. Nevertheless in a real pond, due to the absorption of solar radiation and to the non-constant diffusion coefficients, linear profiles of temperature and salt are rarely present [12].

The influence of radiation absorption in the layer has been previously studied by Giestas *et al.* [4], where the corresponding marginal states were determined. The steady-state solution for temperature confirmed the non-linearity of the profile and the marginal states obtained did not differ appreciably from previous results of Veronis [1, 5], except for the case of low values of the salinity Rayleigh numbers.

The present analysis aims to assess the influence on the stability limits of the joint effect of solar radiation absorption in the layer and non-constant diffusion coefficients. In a solar pond three well-defined zones can be identified (see Fig. 1). The lower and the surface zones are basically convective zones; in the middle zone where gradients are formed convection must be prevented. With this structure solar energy absorbed in the bottom is trapped due to the fact that water is a poor conductor of heat. In the upper layer there is a superposition of free convective motions and of wind driven motion. The lower layer is characterised only by free convective motions or eventually by motions induced by fluid injection or extraction associated with energy extraction from the pond. These two zones have a complex structure and their study is not the aim of this paper. The purpose of the present work is to analyse the stability of the middle zone which is motionless in an initial state due to the presence of the stabilising salt gradient.

The conditions for the non-convectivity of the gradient zone can be analysed assuming a two-dimensional layer of fluid. All dependent variables are functions of the horizontal x and vertical (pointing upwards) z co-ordinates and of the time t. Thermal and salt diffusion coefficients K_T and K_S are considered to be dependent on z. Steady-state solutions are obtained setting the velocity **v** and all time derivatives equal to zero.

The stability of the gradient zone will be analysed by superposing perturbations upon the steady-state solutions and the study of the time evolution of perturbations follows the work of Veronis [1] and Schech-

NOMENCLATURE

a coef. (derived from equation (54))
$A_1(z)$ auxiliary function, equation (51)
$a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$, $c_1(t)$ coef., equations (39)–(43)
C, S_1 coef., equation (63)
C_p specific heat [J kg^{-1} °C^{-1}]
d depth of gradient zone [m]
F coefficient, equation (64)
F_1 coefficent, equation (51)
g acceleration of gravity [m s^{-2}]
h_d convection heat transfer coef. [W m^{-2} °C^{-1}]
k unit vector pointing upwards
$K_T(z)$, $K_S(z)$ thermal and salt diffusivities [m^2 s^{-1}]
K_w thermal conductivity of water [W m^{-1} °C^{-1}]
p pressure [N m^{-2}]
L_1, L_2, L_3 coef., equations (39)–(43)
P coef., equations (58)–(62)
$q(d)$ heat flux at upper boundary ($z = d$) [W m^{-2}]
q heat flux at lower boundary ($z = 0$) [W m^{-2}]
$\dot{q}$ rate of energy generation per unity volume [W m^{-3}]
R_a, R_s Rayleigh numbers for temperature and salinity
S salinity [kg m^{-3}]
t time [s]
T temperature [°C]
T_∞ upper convective zone temperature [°C]
$\mathbf{v}$ velocity field [m s^{-1}]
x, y, z Cartesian co-ordinates.

Greek symbols
α coef. of thermal expansion [°C^{-1}]
β coef. of salt expansion [m^3 kg^{-1}]
λ cell typical length in the x direction [m]
μ extinction coefficient [m^{-1}]
ν kinematic viscosity [m^2 s^{-1}]
ρ, ρ_m density and mean density [kg m^{-3}]
σ_{xz} shear stress [N m^{-2}]
τ inverse Schmidt number
φ_i trial functions
ψ stream function.

Superscripts
x undisturbed variable $\Delta T = T(0) - T(d)$
$\tilde{x}$ perturbed variable $\Delta S = S(0) - S(d)$
$\hat{x}$ adimensionlised variable

$$\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right)$$

$\dot{x}$ time derivative

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

$$J(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}.$$

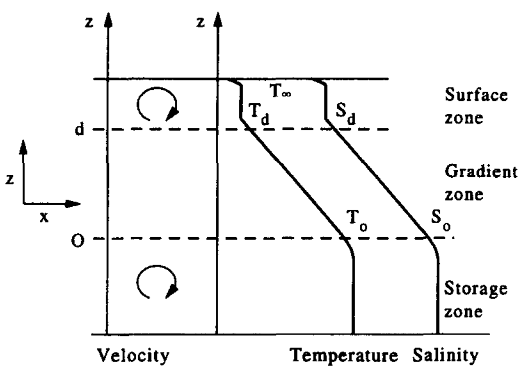


Fig. 1. Solar pond structure.

ter [2]. A weak formulation of the governing equations and a Galerkin method are employed to obtained approximate solutions [9, 10, 6]. This procedure leads to a non-linear system of ordinary differential equations which in order to obtain the linear stability conditions is linearised around the steady-state solution.

2. PROBLEM FORMULATION: THE INFLUENCE OF NON-CONSTANT DIFFUSION COEFFICIENTS

In this section the problem of a layer of fluid with non-constant diffusion coefficients is analysed. To model the gradient zone the Boussinesq approximation of the Navier–Stokes equations [1–4, 7] is written as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho_m}\nabla p + g(\alpha T - \beta S)\vec{k} + \nu\nabla^2\mathbf{v} \quad (1)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (2)$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla \cdot (K_T \nabla T) \quad (3)$$

$$\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = \nabla \cdot (K_S \nabla S) \quad (4)$$

$$\rho = \rho_m(1 - \alpha T + \beta S). \quad (5)$$

The layer is considered to be infinite in the x and y directions and the boundary conditions are of the Dirichlet type: at the upper boundary the temperature takes the value T_d and the salinity the value S_d and at the lower boundary the values T_0 and S_0, respectively.

The boundary values for the variables $\mathbf{v}$, T and S are, see Giestas *et al.* [4]:

$$\mathbf{v}|_{z=d} = 0, \quad \mathbf{v}|_{z=0} = 0 \tag{6}$$

$$\sigma_{xz}|_{z=d} = 0, \quad \sigma_{xz}|_{z=0} = 0 \tag{7}$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=d} = -\frac{h_d}{K_w}(T_d - T_\infty), \quad \left.\frac{\partial T}{\partial z}\right|_{z=0} = -\frac{q}{K_w} \tag{8}$$

$$T|_{z=d} = T_d, \quad T|_{z=0} = T_0 \tag{9}$$

$$S|_{z=d} = S_d, \quad S|_{z=0} = S_0. \tag{10}$$

As can be seen in equation (3) we assume that there is no internal heat generation in the layer of the fluid. The diffusion coefficients dependency will be detailed presently.

2.1. *Diffusion coefficients*

Thermal and salt diffusion coefficients are dependent on both temperature and salt concentration. Nevertheless, the system of equations (1)–(5) considering K_T and K_S to be functions of T and S is very difficult if not impossible to solve analytically. Therefore, we have assumed K_T and K_S to be functions of z, in the same line of thought as Zangrando [8]. Appendix A gives a more detailed account of these dependencies.

In the present analysis a linear variation for K_T and K_S is considered as follows:

$$K_T(z) = K_{T_0} + (K_{T_d} - K_{T_0})\frac{z}{d} \tag{11}$$

$$K_S(z) = K_{S_0} + (K_{S_d} - K_{S_0})\frac{z}{d} \tag{12}$$

where

$$K_{T_0} = K_T(z)|_{z=0}, \quad K_{T_d} = K_T(z)|_{z=d} \tag{13}$$

$$K_{S_0} = K_S(z)|_{z=0}, \quad K_{S_d} = K_S(z)|_{z=d}. \tag{14}$$

2.2. *Steady-state solutions*

Steady state solutions are obtained making the velocity and all time derivatives equal to zero in the set of equations (1)–(5). Substituting K_T and K_S as given by equations (11) and (12) in equations (3) and (4) we obtain:

$$\nabla \cdot (K_T(z)\nabla T) = 0 \tag{15}$$

$$\nabla \cdot (K_S(z)\nabla S) = 0. \tag{16}$$

Solving (15) and (16) and introducing the boundary conditions (13) and (14), the steady-state solutions T_s and S_s are as follows:

$$T_s(z) = T_0 + \frac{T_d - T_0}{\ln(K_{T_d}/K_{T_0})}\ln\left(1 + \left(\frac{K_{T_d}}{K_{T_0}} - 1\right)\frac{z}{d}\right) \tag{17}$$

$$S_s(z) = S_0 + \frac{S_d - S_0}{\ln(K_{S_d}/K_{S_0})}\ln\left(1 + \left(\frac{K_{S_d}}{K_{S_0}} - 1\right)\frac{z}{d}\right). \tag{18}$$

Figure 2 depicts these profiles for the following typical values:

$$\begin{aligned}
&d = 1\ \text{m} \\
&T_d = 20^\circ\text{C}, \quad S_d = 0\% \\
&T_0 = 100^\circ\text{C}, \quad S_0 = 20\% \\
&K_{T_d} = 1.43 \times 10^{-7}\ \text{m}^2\ \text{s}^{-1}, \quad K_{S_d} = 1.39 \times 10^{-9}\ \text{m}^2\ \text{s}^{-1} \\
&K_{T_0} = 1.73 \times 10^{-7}\ \text{m}^2\ \text{s}^{-1}, \quad K_{S_0} = 5.94 \times 10^{-9}\ \text{m}^2\ \text{s}^{-1}.
\end{aligned} \tag{19}$$

The figure clearly shows the nonlinearity of the salinity profile to be greater than the nonlinearity of the temperature profile, a direct consequence of the diffusivity of temperature being greater than the diffusivity of salt. We may note that Veronis [1] linear profiles are easily recovered from equations (17) and (18) by making $K_{T_d} \to K_{T_0}$ and $K_{S_d} \to K_{S_0}$, as expected.

2.3. *Approximate formulation*

To satisfy equation (2) identically, the velocity $\mathbf{v}$ is expressed in terms of the stream function ψ defined by:

$$\mathbf{v} = (u, v, w) = \left(\frac{\partial \psi}{\partial z}, 0, -\frac{\partial \psi}{\partial x}\right). \tag{20}$$

The dependent variables T, S and ψ are considered to be the sum of the steady-state solutions T_s, S_s and $\psi_s (\equiv 0)$ and of the perturbation terms $\tilde{T}$, $\tilde{S}$ and $\tilde{\psi}$ as follows:

$$T = T_s + \tilde{T} \tag{21}$$

$$S = S_s + \tilde{S} \tag{22}$$

$$\psi = \tilde{\psi}. \tag{23}$$

A non-dimensionlisation is performed in accordance with [1]. Diffusivities K_T and K_S are non-dimensionalised by their average values $\bar{K}_T$ and $\bar{K}_S$ defined, respectively, as:

$$\bar{K}_T = \frac{1}{d}\int_0^d K_T(z)\,dz \tag{24}$$

$$\bar{K}_S = \frac{1}{d}\int_0^d K_S(z)\,dz \tag{25}$$

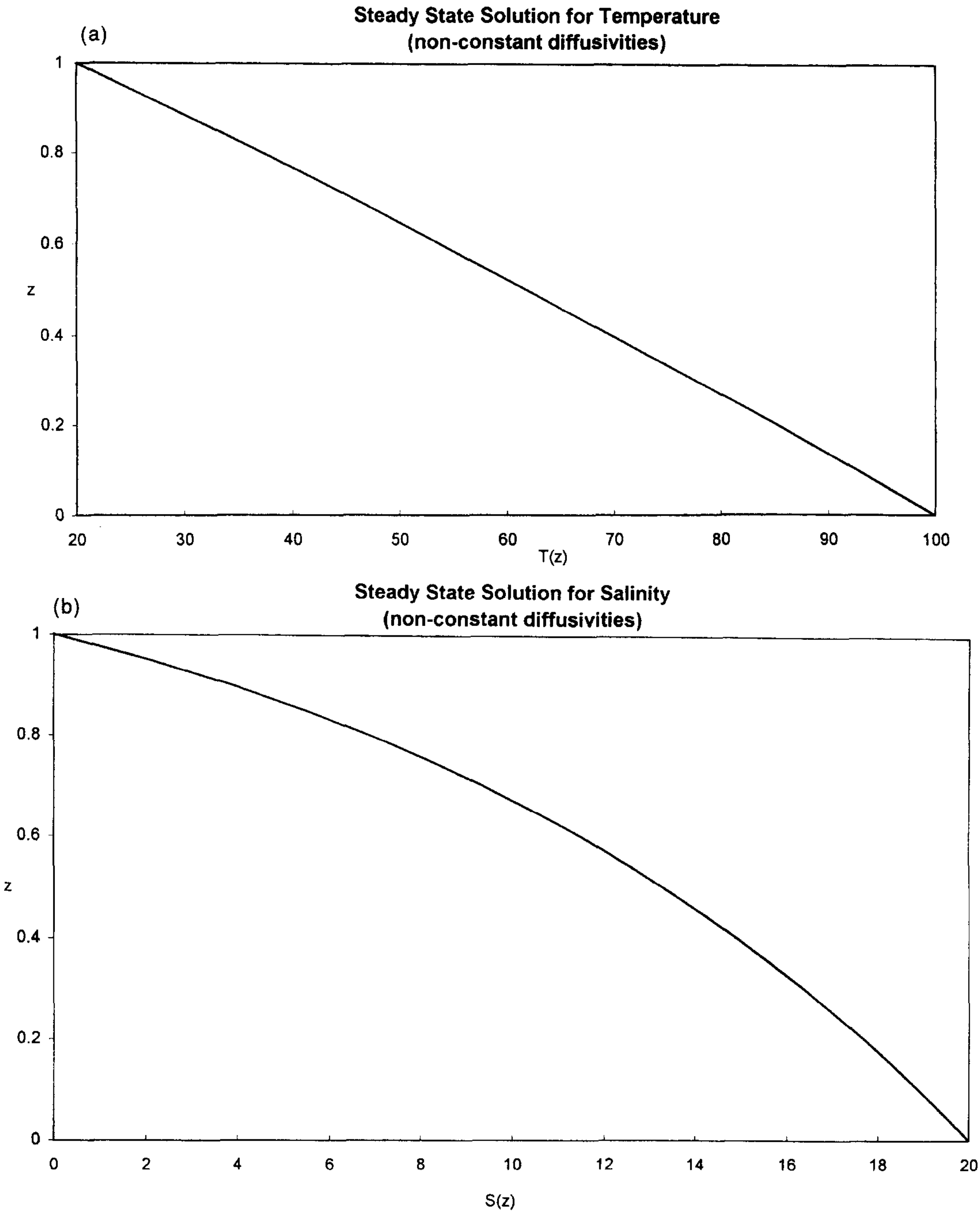


Fig. 2. (a) Steady-state temperature (non-constant diffusivities); (b) steady-state salinity (non-constant diffusivities).

thus becoming

$$\hat{K}_T = \frac{K_T}{\bar{K}_T}, \quad \hat{K}_S = \frac{K_S}{\bar{K}_S}. \tag{26}$$

Inserting relations (21)–(23) in the system of equations (1)–(5) and taking into consideration the above non-dimensionalisation, the following set of equations is obtained:

$$\frac{\partial \hat{T}}{\partial \hat{t}} - J(\hat{\psi}, \hat{T}) + f(z)\frac{\partial \hat{\psi}}{\partial \hat{x}} = \hat{\nabla}\cdot(\hat{K}_T(z)\hat{\nabla}\hat{T}) \tag{27}$$

$$\frac{\partial \hat{S}}{\partial \hat{t}} - J(\hat{\psi}, \hat{S}) + g(z)\frac{\partial \hat{\psi}}{\partial \hat{x}} = \frac{\bar{K}_T}{\bar{K}_S}\hat{\nabla}\cdot(\hat{K}_S(z)\hat{\nabla}\hat{S}) \tag{28}$$

$$\left(\frac{1}{\overline{Pr}}\frac{\partial}{\partial \hat{t}} - \hat{\nabla}^2\right)\hat{\nabla}^2\hat{\psi} = -\bar{R}_a\frac{\partial \hat{T}}{\partial \hat{x}} + \bar{R}_s\frac{\partial \hat{S}}{\partial \hat{x}} + \frac{1}{\overline{Pr}}J(\hat{\psi}, \hat{\nabla}^2\hat{\psi}) \tag{29}$$

with

$$f(z) = \frac{K_{T_d} - K_{T_0}}{\ln(K_{T_d}/K_{T_0})(K_{T_0} - (K_{T_d} - K_{T_0})\hat{z})} \tag{30}$$

$$g(z) = \frac{K_{S_d} - K_{S_0}}{\ln(K_{S_d}/K_{S_0})(K_{S_0} - (K_{S_d} - K_{S_0})\hat{z})}. \quad (31)$$

The thermal and salinity Rayleigh numbers, $\bar{R}_a$ and $\bar{R}_s$, respectively, and the Prandtl number $\overline{Pr}$ are also obtained using the average values for the thermal diffusion coefficients:

$$\bar{R}_a = \frac{g\alpha\Delta T d^3}{\bar{K}_T \nu}, \quad \bar{R}_s = \frac{g\beta\Delta S d^3}{\bar{R}_T \nu}, \quad \overline{Pr} = \frac{\nu}{\bar{K}_T}. \quad (32)$$

The set of non-dimensionalised equations (27)–(29) is recast into a weak formulation, cf. [4] and $\tilde{S}$, $\tilde{T}$ and $\tilde{\psi}$ are approximated by the following linear combination of functions:

$$\hat{T} = a_1(t)\varphi_1(x,z) + a_2(t)\varphi_2(x,z) \quad (33)$$

$$\hat{S} = b_1(t)\varphi_1(x,z) + b_2(t)\varphi_2(x,z) \quad (34)$$

$$\hat{\psi} = c_1(t)\varphi_3(x,z). \quad (35)$$

where φ_1, φ_2 and φ_3 are chosen as explained in [4] as:

$$\varphi_1(x,z) = z(1-z) \quad (36)$$

$$\varphi_2(x,z) = z(1-z)\cos\left(\frac{2\pi x}{\lambda}\right) \quad (37)$$

$$\varphi_3(x,z) = z^2(1-z)\sin\left(\frac{2\pi x}{\lambda}\right). \quad (38)$$

Introducing the approximations (33)–(35) in the weak formulation of (27)–(29) as in [4] the following system of ordinary differential equations for $a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$ and $c_1(t)$ is obtained:

$$\dot{a}_1 = -10a_1 + \frac{\pi}{14\lambda}c_1 a_2 \quad (39)$$

$$\dot{a}_2 = -\frac{2}{\lambda}L_2 a_2 - \frac{\pi}{\lambda}\left[C + \frac{a_1}{7}\right]c_1 \quad (40)$$

$$\dot{b}_1 = -10\frac{\bar{K}_S}{\bar{K}_T}b_1 + \frac{\pi}{14\lambda}c_1 b_2 \quad (41)$$

$$\dot{b}_2 = -\frac{2}{\lambda}\frac{\bar{K}_S}{\bar{K}_T}L_2 b_2 - \frac{\pi}{\lambda}\left[S_1 + \frac{b_1}{7}\right]c_1 \quad (42)$$

$$\dot{c}_1 = -2\,\overline{Pr}\,L_1 L_3 c_1 - \frac{7}{4}\pi\,\overline{Pr}\,L_1\lambda(\bar{R}_a a_2 - \bar{R}_s b_2) \quad (43)$$

where

$$L_1 = \frac{1}{2\pi^2 + 7\lambda^2}, \quad L_2 = \frac{2\pi^2 + 5\lambda^2}{\lambda},$$

$$L_3 = \frac{4\pi^4 + 28\lambda^2\pi^2 + 105\lambda^4}{\lambda^2} \quad (44)$$

and where C and S_1 depend on the constants K_{T_0}, K_{T_d}, $\bar{K}_T$ and K_{S_0}, K_{S_d} and $\bar{K}_S$ as presented in Appendix B.

3. PROBLEM FORMULATION: THE INFLUENCE OF BOTH SOLAR RADIATION ABSORPTION AND NON-CONSTANT DIFFUSION COEFFICIENTS

In this section we deal with the problem of a layer of fluid considering the effect of solar radiation absorption and assuming non-constant temperature and salt diffusion coefficients. Therefore, the dependence of K_T and K_S on z as given by equations (11) and (12) and the solution for salinity as proposed in equation (18) are assumed.

We may note that the fact of considering K_T and K_S as functions of z instead of T and S allows the decoupling of the steady-state solutions of temperature and salinity.

In order to take into account the effect of solar radiation absorption equation (3) is replaced by:

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \nabla\cdot(K_T\nabla T) + \frac{\dot{q}}{\rho C_p} \quad (45)$$

where $\dot{q}$ is the rate of energy generation per unit volume in the layer and is given by (see [4]):

$$\dot{q}(z) = q(d)\mu\exp(-\mu(d-z)) \quad (46)$$

where $q(d)$ is the heat flux due to solar radiation at the upper boundary of the gradient zone and μ is the extinction coefficient. The steady-state heat diffusion equation is derived from (45) making the velocity and all time derivatives equal to zero obtaining:

$$\frac{\partial T_s}{\partial z} = -\frac{q(z)}{K_T(z)\rho C_p} + \frac{\mathscr{C}}{K_T(z)} \quad (47)$$

where $\mathscr{C}$ is a constant of integration.

Considering the following flux boundary conditions for temperature:

$$\left.\frac{\partial T_s}{\partial z}\right|_{z=d} = -\frac{h_d(T_d - T_\infty)}{K_{T_d}\rho C_p} \quad (48)$$

$$\left.\frac{\partial T_s}{\partial z}\right|_{z=0} = -\frac{q}{K_{T_0}\rho C_p} \quad (49)$$

where q is the total heat flux absorbed in the bottom convective zone which can be written according to [4] as:

$$q = (1-f_e)q(d)\exp(-\mu d) \quad (50)$$

and f_e denotes the heat flux extracted at the bottom expressed as a fraction of the total heat flux absorbed

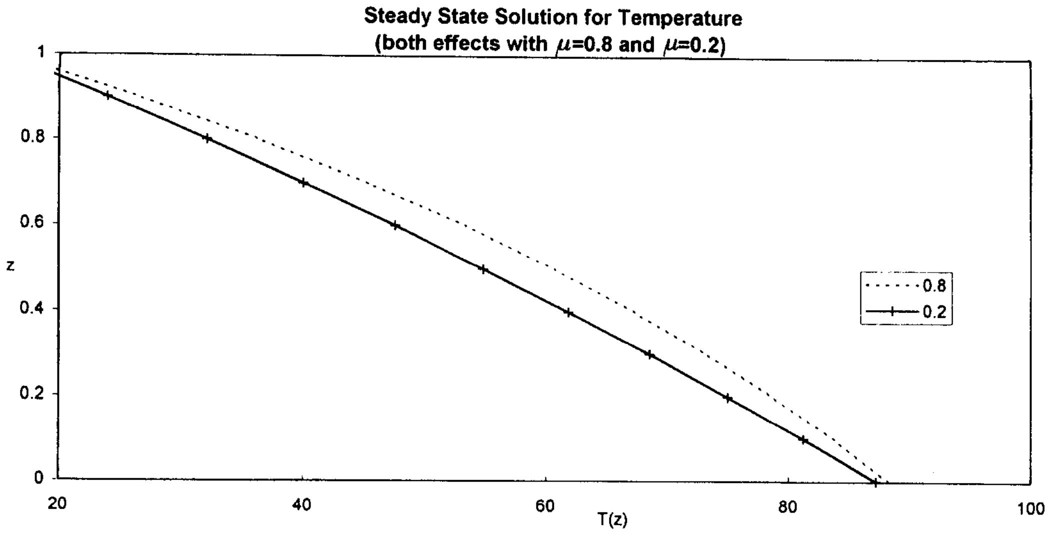


Fig. 3. Steady-state temperature considering the effects of non-constant diffusivities and solar radiation absorption with $\mu = 0.8$ and $\mu = 0.2$.

in the bottom convective zone, the steady-state solution for temperature is obtained as:

$$T_s(z) = T_\infty - F_1 \exp\left(\frac{-K_{T_0}\mu d}{K_{T_d}-K_{T_0}}\right) \times [Ei(A_1(z)) - Ei(A_1(d))] + f_e F_1 \ln\left(1+\frac{K_{T_0}}{K_{T_d}}\left(1-\frac{z}{d}\right)\right) + \frac{q}{h_d}(1-\exp(-\mu d) f_e) \quad (51)$$

with

$$F_1 = \frac{gd\exp(-\mu d)}{\rho C_p (K_{T_d}-K_{T_0})}, \quad A_1(z) = \mu\left(z+\frac{K_{T_0}d}{K_{T_d}-K_{T_0}}\right) \quad (52)$$

where $Ei(x)$ is the exponential integral function defined by:

$$Ei(x) = \int_{-\infty}^{x} \frac{\exp(t)}{t}\,dt. \quad (53)$$

A graphical representation of the steady-state solution obtained for temperature is presented in Fig. 3 where the effect of solar radiation absorption on the profile curvature is clear.

Making $K_{T_d} \to K_{T_0}$ and $K_{S_d} \to K_{S_0}$ in equation (51) the correspondent steady-state solution with constant diffusion coefficients presented in [4] is easily recovered.

Taking into account the steady-state solutions equations (18) and (51) and the respective boundary conditions, the set of equations (1)–(5) with equation (3) replaced by equation (45) becomes, after non-dimensionalisation:

$$\frac{\partial \hat{T}}{\partial \hat{t}} - J(\hat{\psi}, \hat{T}) + \hat{a}\left(\frac{\exp(\hat{\mu}\hat{z}) - f_e}{\hat{K}_{T_0} + (\hat{K}_{T_d} - \hat{K}_{T_0})\hat{z}}\right)\frac{\partial \hat{\psi}}{\partial \hat{x}} = \hat{\nabla}\cdot(\hat{K}_T(z)\hat{\nabla}\hat{T}) \quad (54)$$

$$\frac{\partial \hat{S}}{\partial \hat{t}} - J(\hat{\psi}, \hat{S}) + g(z)\frac{\partial \hat{\psi}}{\partial \hat{x}} = \frac{\bar{K}_T}{\bar{K}_S}\hat{\nabla}\cdot(\hat{K}_S(z)\hat{\nabla}\hat{S}) \quad (55)$$

$$\left(\frac{1}{\overline{Pr}}\frac{\partial}{\partial \hat{t}} - \hat{\nabla}^2\right)\hat{\nabla}^2\hat{\psi} = -\bar{R}_a\frac{\partial \hat{T}}{\partial \hat{x}} + \bar{R}_s\frac{\partial \hat{S}}{\partial \hat{x}} + \frac{1}{\overline{Pr}}J(\hat{\psi}, \hat{\nabla}^2\hat{\psi}) \quad (56)$$

with:

$$\hat{a} = \frac{d}{\Delta T}a, \quad a = \frac{-gd\exp(-\mu d)}{\rho C_p}. \quad (57)$$

Considering the same approximation functions as in the previous section system (39)–(43) becomes now:

$$\dot{a}_1 = -10a_1 + \frac{\pi}{14\lambda}c_1 a_2 \quad (58)$$

$$\dot{a}_2 = -\frac{2}{\lambda}L_2 a_2 - \frac{\pi}{\lambda}\left[\hat{a}P + \frac{a_1}{7}\right]c_1 \quad (59)$$

$$\dot{b}_1 = -10\frac{\bar{K}_S}{\bar{K}_T}b_1 + \frac{\pi}{14\lambda}c_1 b_2 \quad (60)$$

$$\dot{b}_2 = -\frac{2}{\lambda}\frac{\bar{K}_S}{\bar{K}_T}L_2 b_2 - \frac{\pi}{\lambda}\left[S_1 + \frac{b_1}{7}\right]c_1 \quad (61)$$

$$\dot{c}_1 = -2\overline{Pr}\,L_1 L_3 c_1 - \frac{7}{4}\pi\overline{Pr}\,L_1\lambda(\bar{R}_a a_2 - \bar{R}_s b_2) \quad (62)$$

where P denotes a polynomial in K_{T_0}, K_{T_d} and μ. The expression for this polynomial is very long and of no particular relevance and, thus, is not included in this paper, but is available from the authors, upon request.

4. STABILITY RESULTS

Studies concerning stability criteria for a layer of fluid with linear profiles of temperature and salt were made by Veronis [1] and Schechter [2], employing a linear theory to obtain the marginal states for stability. In the present analysis the same procedure is adopted.

Systems (39)–(43) and (58)–(62) are linearised about the origin and solutions of the type $A_i \exp(k\omega t)$ are proposed leading to a linear system of algebraic equations. The characteristic polynomial associated with the matrix of this system provides the stability conditions for the problem [4]. These conditions determine the different zones in the (R_s, R_a) plane which correspond to different types of motion.

The transition from a pure conductive state to a steady convective state is made through an unstable motion beginning with oscillatory motions [1]. The results obtained for the stability marginal states for the different cases are summarised as follows.

Case 1: *The influence of non-constant diffusion coefficients*

The marginal states for stability obtained from system (39)–(43) are:

(1) Marginal state equation for steady convective motion

$$\bar{R}_a = \left(\frac{S_1}{C}\right)\left(\frac{1}{\bar{\tau}}\right)\bar{R}_a + \frac{16}{7}\frac{L_3 L_2}{\pi^2 C\lambda}. \quad (63)$$

(2) Marginal state equation for the onset of oscillatory motion

$$\bar{R}_a = \left(\frac{S_1}{C}\right)\left(\frac{\bar{\tau}F+\overline{Pr}}{F+\overline{Pr}}\right)\bar{R}_s + \frac{16}{7}\frac{L_2(\bar{\tau}+1)[\bar{\tau}F+\overline{Pr}]}{\pi^2 C\,\overline{Pr}\,\lambda^2 L_1} \quad (64)$$

where

$$\bar{\tau} = \frac{\bar{K}_S}{\bar{K}_T}, \quad F = \frac{L_2}{\lambda L_1 L_3}. \quad (65)$$

In the present case the following typical values for a solar pond are used:

$$\bar{K}_S = 3.665\times10^{-9}\ \mathrm{m^2\,s^{-1}}, \quad \bar{K}_T = 1.58\times10^{-7}\ \mathrm{m^2\,s^{-1}}$$
$$\overline{Pr} = 4.11, \quad \bar{\tau} = 0.02319. \quad (66)$$

Introducing these data and $\lambda = 2.131$ obtained as described in [4], equations (63) and (64) become:

$$\bar{R}_a = 42.49\bar{R}_s + 1795.8 \quad (67)$$

$$\bar{R}_a = 0.868\bar{R}_s + 1843.4. \quad (68)$$

The zone in the (R_s, R_a) plane where oscillatory motion takes place is very narrow and is delimited by the intersection of the half planes:

$$\bar{R}_a > 0.868\bar{R}_s + 1843.4 \quad (69)$$

$$\bar{R}_a < 0.985\bar{R}_s + 1843.2 \quad (70)$$

as shown in Fig. 4. This points to the conclusion that there is practically no transition to an unstable state through periodic motions.

Case 2: *The joint effect of the influence of solar radiation absorption and non-constant diffusivities*

The marginal states for stability obtained from system (58)–(62) are:

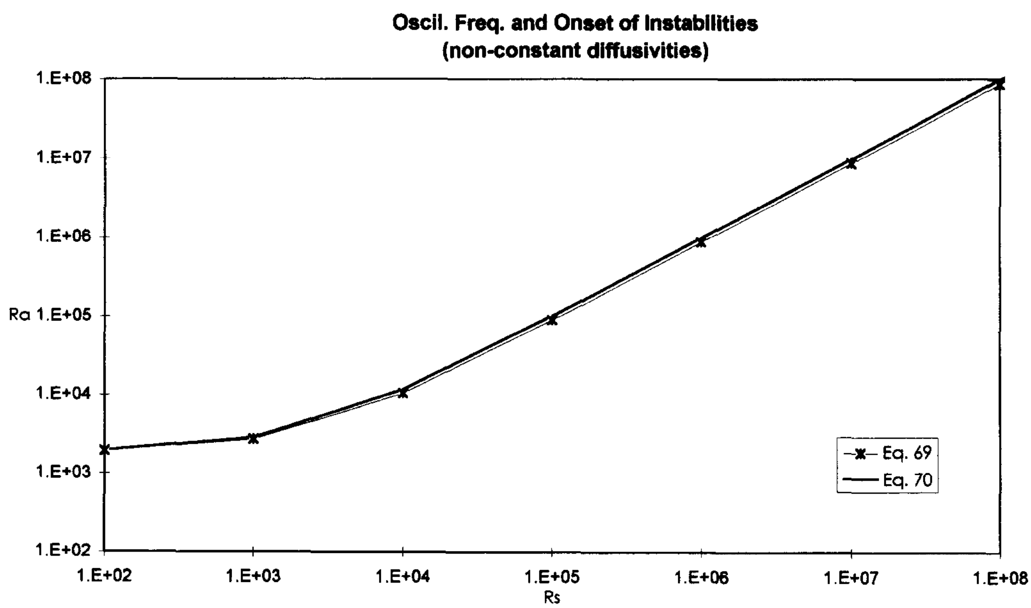


Fig. 4. Oscillatory frequency and onset of oscillatory motion (non-constant diffusivities).

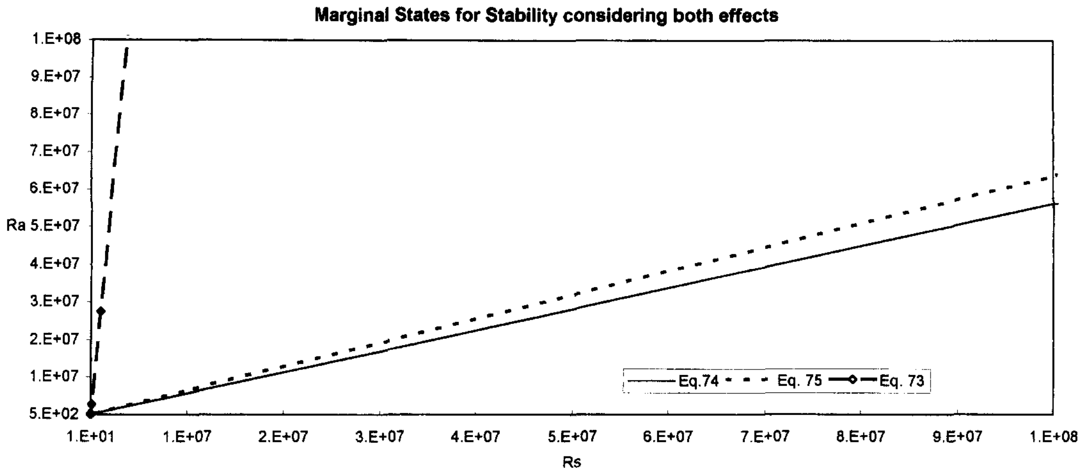


Fig. 5. Marginal states for stability considering the effects of non-constant diffusivities and solar radiation absorption with $\mu = 0.8$.

(1) Marginal state equation for steady convective motion

$$\bar{R}_a = \left(\frac{S_1}{\hat{a}P}\right)\left(\frac{1}{\bar{\tau}}\right)\bar{R}_s + \frac{16}{7}\left(\frac{1}{\hat{a}P}\right)\frac{L_3L_2}{\pi^2\lambda}. \quad (71)$$

(2) Marginal state equation for the onset of oscillatory motion

$$\bar{R}_a = \left(\frac{S_1}{\hat{a}P}\right)\left(\frac{\bar{\tau}F+\overline{Pr}}{F+\overline{Pr}}\right)\bar{R}_s + \frac{16}{7}\left(\frac{1}{\hat{a}P}\right)\frac{L_2(\bar{\tau}+1)[\bar{\tau}F+\overline{Pr}]}{\pi^2\,\overline{Pr}\,\lambda^2 L_1}. \quad (72)$$

The dependence on salt diffusion coefficient is conveyed in S_1 and the effect of considering solar radiation absorption is expressed in the factor $1/\hat{a}P$ which is a function of K_{T_0}, K_{T_d} and μ.

Employing the same values as in Case 1 and $\mu = 0.8$, equations (71) and (72), become:

$$\bar{R}_a = 27.5\bar{R}_s + 1160.9 \quad (73)$$

$$\bar{R}_a = 0.562\bar{R}_s + 34.35. \quad (74)$$

The half plane that limits oscillatory motion is given now by:

$$\bar{R}_a < 0.63\bar{R}_s + 1191.5. \quad (75)$$

Figure 5 depicts equations (73) through (75).

The most evident conclusion is that these relations lead to a stability situation clearly more restrictive than the case corresponding to non-constant diffusion coefficients only. It can be seen from Fig. 5 that oscillatory motions occur for a wider range of values of $\bar{R}_a$.

4.1. *Comparison with previous studies*

Veronis [1], Schechter [2] and Turner [3] provided the basis for stability studies concerning double diffusion systems as, for instance, thermohaline convection in solar ponds. Table 1 summarises the results obtained in the present analysis, the results obtained previously in [4] and those obtained earlier by Veronis [1]. The results are presented in terms of the slopes of the marginal states equations. Note that as the Rayleigh numbers for solar ponds are in the order of 10^9 the independent terms of these equations can be neglected. From Table 1 we may conclude that for the onset of instabilities Veronis stability results are more restrictive than those prevailing when the effects of solar radiation absorption and of non-constant diffusivities are considered in a decoupled way, but less restrictive when these effects are considered together.

Figure 6 shows the equation for the onset of instabilities obtained by Veronis and the equation for the onset of instabilities when the effect of non-constant diffusivities, equation (68), is taken into account. For typical values of the Rayleigh numbers the results are very similar, the greatest difference occurring only for small Rayleigh numbers values, that is, for a situation of a layer with a low salinity gradient or with a high viscosity.

Figure 7 represents the equation for the onset of instabilities for the case of non-constant diffusivities, the case of both effects and Veronis results. For the onset of steady convective motion we may conclude that Veronis results are less restrictive than the results of considering the effect of solar radiation absorption and the effect of non-constant diffusivities either coupled or decoupled, as can be seen in Fig. 8. As a general conclusion we may say that, when the joint effects of solar radiation absorption and of non-constant thermal and salt concentration diffusion coefficients are present, the marginal states for the beginning of instabilities occurs for smaller values of the thermal Rayleigh number.

Table 1. Slopes of marginal states equations

	(1)	(2)	(3)	(4)
(i)	$\bar{R}_a = 0.809\bar{R}_s$	$\bar{R}_a = 0.845\bar{R}_s$	$\bar{R}_a = 0.868\bar{R}_s$	$\bar{R}_a = 0.562\bar{R}_s$
(ii)	$\bar{R}_a = 43.16\bar{R}_s$	$\bar{R}_a = 42.49\bar{R}_s$	$\bar{R}_a = 41.44\bar{R}_s$	$\bar{R}_a = 27.52\bar{R}_s$

(1) Veronis results [1].
(2) Results obtained when the effect of solar radiation absorption is considered [4].
(3) Results obtained when the effect of non-constant diffusivities for temperature and salt is considered.
(4) Results obtained when both effects are present.
(i) Onset of instabilities.
(ii) Onset of steady convective motion.

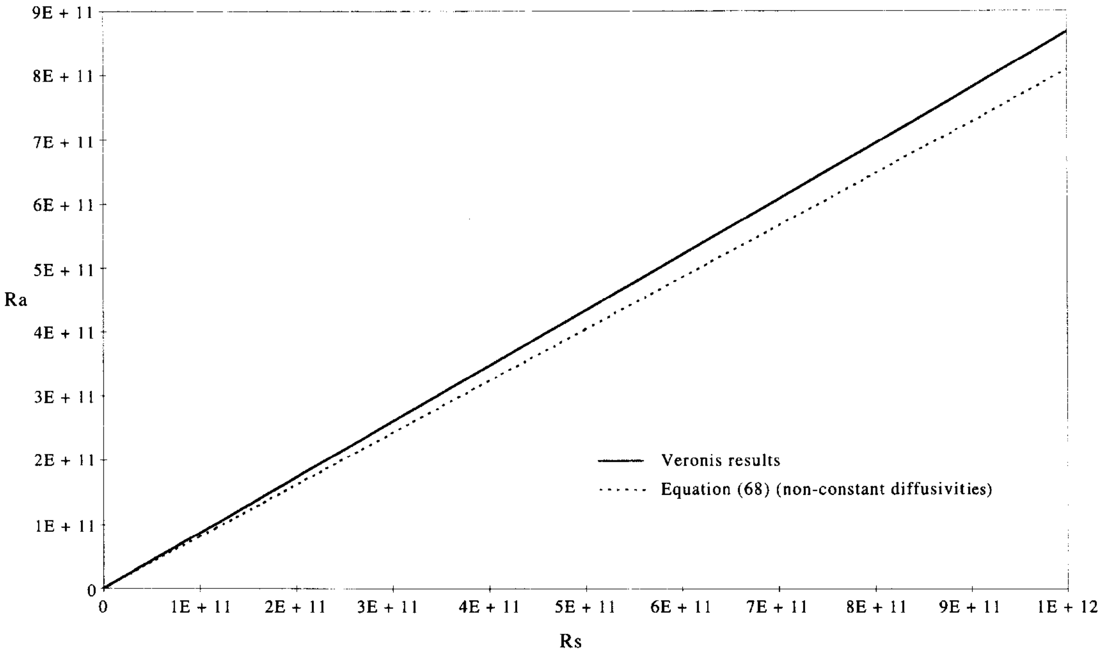


Fig. 6. Comparison of onset of oscillatory motion considering non-constant diffusivities and Veronis analysis results.

5. CONCLUSIONS

The thermohaline problem has been extensively studied by different authors [1–3]. Linear profiles of salt and temperature were generally assumed and stability criteria for this situation have been determined. The double diffusion problem was previously [1, 2, 7] studied considering a layer of fluid heated from below and with constant thermal and salt diffusion coefficients leading to linear profiles for both temperature and salt. In these conditions the stability criteria for the layer was determined in terms of thermal and solutal Rayleigh numbers [1]. The results obtained have been extensively used in the context of solar ponds in order to determine the gradient of salinity needed for preventing convection.

Nevertheless, in the gradient zone of a solar pond linear profiles of both salt and temperature are rarely present due to solar radiation absorption in the layer and to non-constant diffusion coefficients.

The problem of stability of a layer of fluid with linear salinity profile and with heat absorption in the layer, leading to a non-linear temperature profile, was addressed in [4]. The results obtained showed that the marginal state condition for the stability of the gradient zone (initiation of oscillatory motion) was less restrictive than those obtained by Veronis [1] for the case of linear profiles.

In the present paper we have considered: (i) the influence of non-constant diffusion coefficients for both heat and salt yielding non-linear profiles for temperature and salt; and (ii) the problem when the effects of non-constant diffusion coefficients and heat absorption in the layer are taken together.

The analysis showed that the temperature profile is basically controlled by the solar radiation absorption

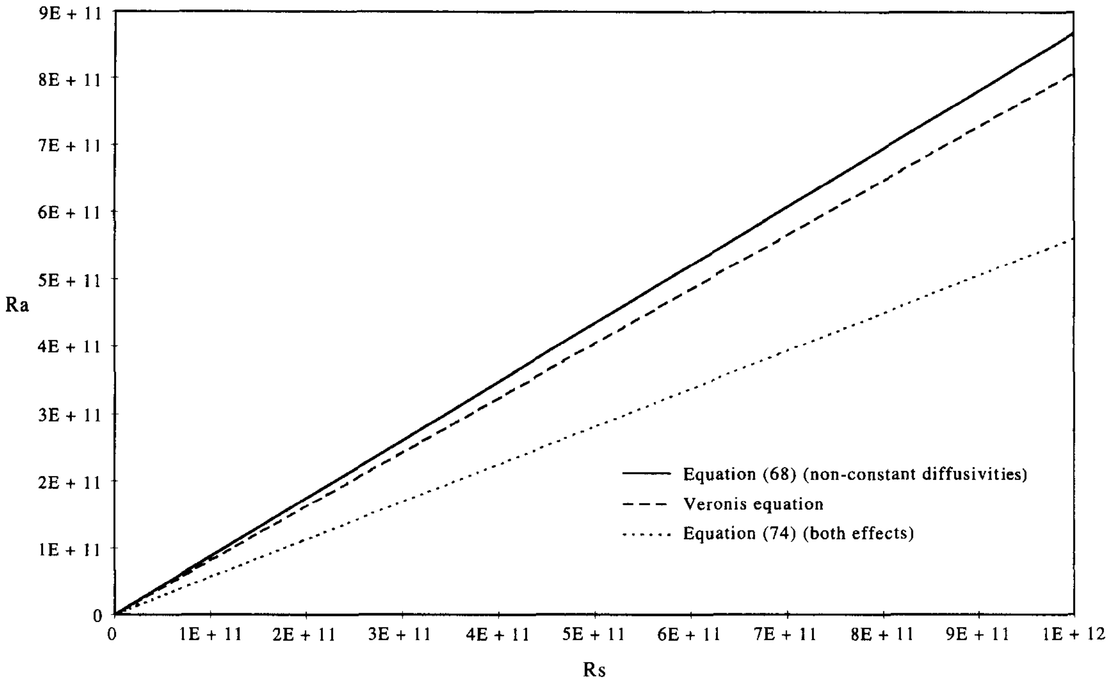


Fig. 7. Marginal states for the onset of oscillatory motion considering the effects of non-constant diffusivities and solar radiation absorption and Veronis criteria.

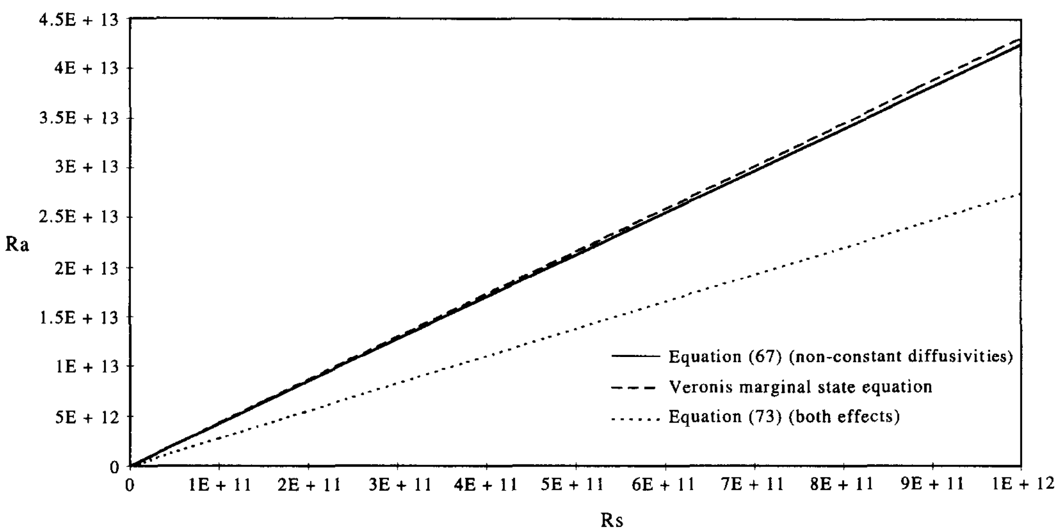


Fig. 8. Marginal states for the steady convection considering the effects of non-constant diffusivities and solar radiation absorption and Veronis criteria.

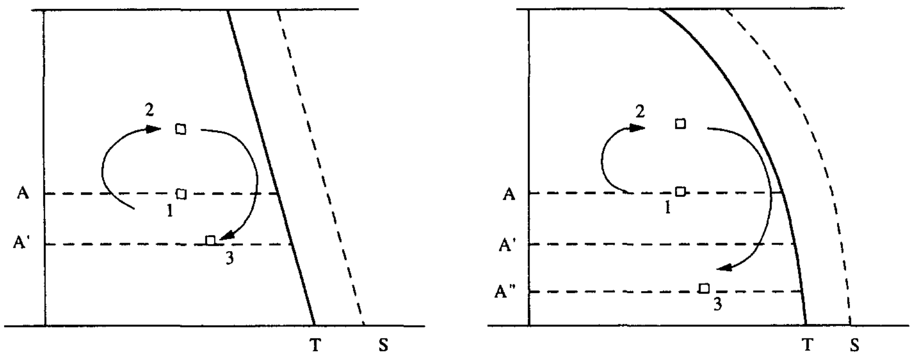
in the layer, whereas the salinity profile is a consequence of non-constant diffusion coefficients of both salt and temperature.

When non-constant diffusion coefficients and solar radiation absorption are taken separately the marginal states obtained differ slightly from the results obtained by Veronis. However, when considering both effects the results differ appreciably and are much more restrictive, that is, the onset of oscillatory motions starts at values of R_a lower than the critical values of R_a derived by Veronis.

This result is a consequence of the ratio between thermal and salinity diffusion coefficients being about two times greater than when those effects were considered in a decoupled form. This situation can be analysed with the help of Fig. 9 where the motion of an hypothetical fluid element is described. In this figure, two different situations are depicted.

The first considers linear profiles for both temperature and salt and reflects the analysis performed by Veronis. In a second situation both profiles are non-linear as a consequence of solar radiation absorption and non-constant diffusion coefficients.

In both situations a given fluid element starts at



Linear profiles for both temperature and salinity

Non-linear profiles for both temperature and salinity (case of both effects)

Fig. 9. Diagram of the onset of instabilities considering the transport of a fluid element.

position A and rises in the layer as a consequence of a small perturbation. In the first situation, the element reaches a less warm and with less salt concentration zone (2), becoming heavier and falling to a position (3) below the starting position (1). This shows how a small perturbation can be amplified as a consequence of the difference between thermal and salt diffusivities.

In the second situation the same fluid element reaches the same less warm zone (2) than in previous situation. In this case, the element loses less salt and becomes heavier falling to a place A'' with $A'' < A$. This figure shows that in this situation small perturbations are more amplified than in the previous case. Therefore, instabilities take place for smaller values of thermal Rayleigh numbers.

It remains clear that to maintain the non-convectivity of the layer of fluid, i.e. to maintain the layer stable it is necessary to increase the gradient of salinity in the pond and, therefore, more salt is needed to prevent instabilities than the amount considered previously.

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APPENDIX A: DIFFUSION COEFFICIENTS DEPENDENCIES

Thermal and salt diffusion coefficients are dependent on both temperature and salt concentration. A least square fit to the International Critical Tables for a range of temperature between 5 and 100°C and for a salinity range between 0 and 20% yield the following expressions for K_T and K_S:

$$K_T = (1.33+0.00563T-0.00002T^2 +0.00603S_w-0.00019S_w^2)\times 10^{-7} \quad \text{(A1)}$$

$$K_S = (8.16+0.255T+0.00254T^2-0.28S_w +0.0147S_w^2)\times 10^{-10} \quad \text{(A2)}$$

where S_w denotes the value of salt concentration in weight percent, $S_w = S/\rho \times 100$.

K_T and K_S can be considered as functions of the depth z. The relations (15) and (16) can, thus, be written as:

$$K_T(T,S)\frac{\partial T}{\partial z} = C_1 \quad \text{(A3)}$$

$$K_S(T,S)\frac{\partial S}{\partial z} = C_2 \quad \text{(A4)}$$

where C_1 and C_2 are integration constants. The solutions for K_T and K_S as functions of z were numerically obtained

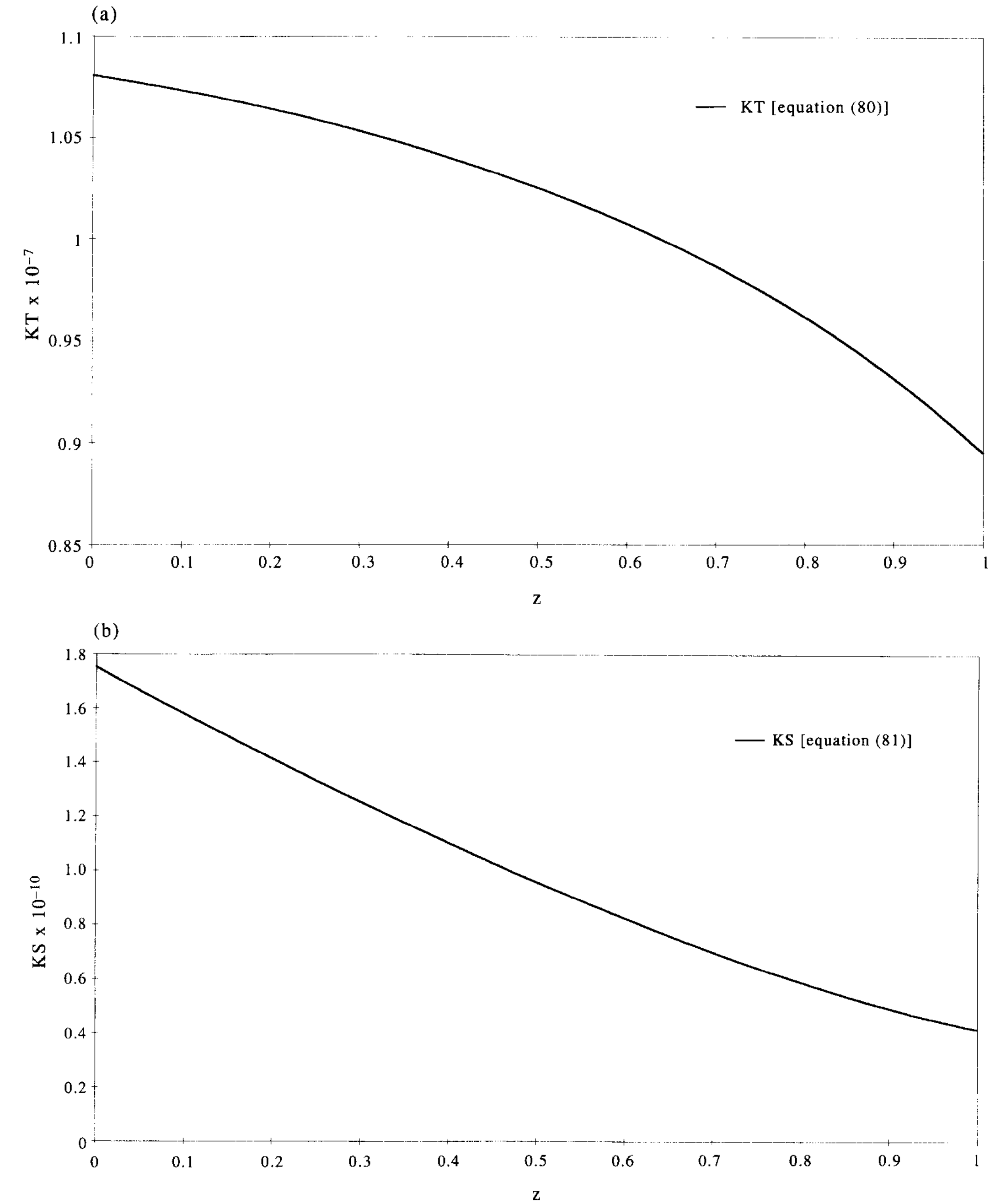

Fig. A1. (a) K_T as function of z; (b) K_S as function of z.

through a 4th order Runge–Kutta method together with a shooting procedure to adjust the integration constants, in order to satisfy the boundary conditions values for T and for S. The relations obtained are:

$$K_T = 1.7319473 \times 10^{-7} - 4.740486 \times 10^{-9} z - 2.3631973 \times 10^{-8} z^2 \quad (A5)$$

$$K_S = (5.9608745 - 6.250911 z + 1.652486 z^2) \times 10^{-9} \quad (A6)$$

and are depicted in Fig. A1.

The steady-state solutions for both temperature and salinity were derived from equations (15) and (16) assuming relations (A5) and (A6), yielding:

$$T_s = -203.0656249 \operatorname{arctanh}(0.3691346365 z + 0.03702351844) + 107.5216419 \quad (A7)$$

$$S_s = -198.9594401 \arctan(5.778149926 z - 10.92859514) - 274.3619146. \quad (A8)$$

These profiles are practically the same as those of equations (17) and (18), obtained analytically, when K_T and K_S are considered as linear functions of z, as can be easily seen in Fig. A2.

This result validates the basic assumption that thermal and salt concentration diffusion coefficients can be considered as linear functions of z. We may note that in equations (17) and

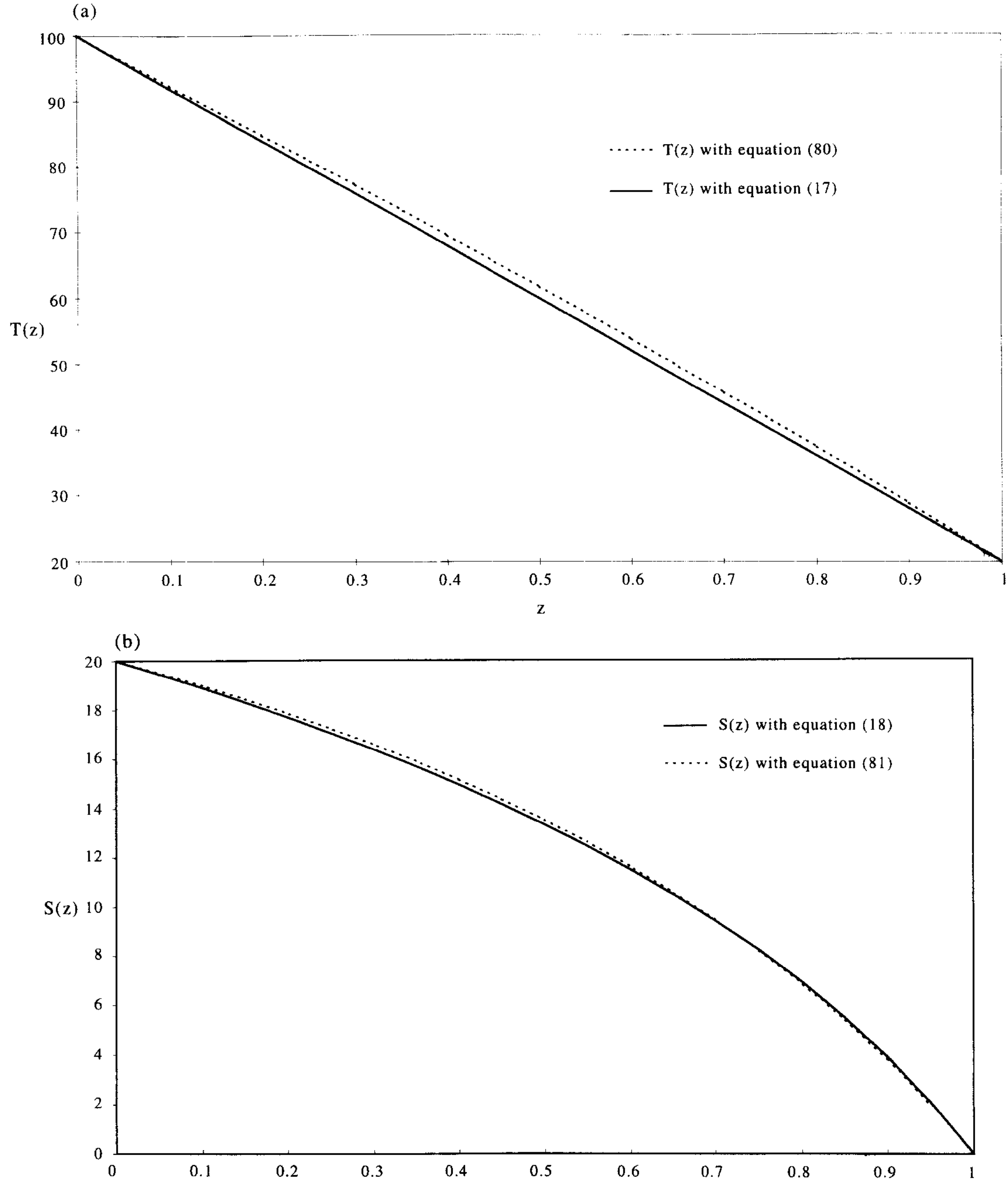

Fig. A2. (a) Steady-state temperature considering K_T as a quadratic function of z; (b) steady-state salinity considering K_S as a quadratic function of z.

(18) the contribution of the different physical parameters is clear which is not the case when numeric equations (A7) and (A8) are considered.

APPENDIX B: EXPRESSION FOR THE COEFFICIENTS C AND S_1 APPEARING IN EQUATIONS (39)–(43)

$$C = (-15K_{T_d}^4 K_{T_0} - 20K_{T_d}^2 k_{T_0}^3 - 30K_{T_d} K_{T_0}^4 - 60K_{T_d}^2 K_{T_0} \ln(K_{T_d})$$
$$+3K_{T_0}^5 + 60K_{T_d}^3 K_{T_0}^2 + 2K_{T_d}^5 + 60K_{T_d}^2 K_{T_0}^3 \ln(K_{T_0}))/$$
$$(\ln(K_{T_d} - \ln(K_{T_0}))(K_{T_0} - K_{T_d})^5 \quad \text{(B1)}$$

$$S_1 = (-15K_{S_d}^4 K_{S_0} - 20K_{S_d}^2 K_{S_0}^3 - 30K_{S_d} K_{S_0}^4 - 60K_{S_d}^2 K_{S_0} \ln(K_{S_d})$$
$$+3K_{S_0}^5 + 60K_{S_d}^3 K_{S_0}^2 + 2K_{S_d}^5 + 60K_{S_d}^2 K_{S_0}^3 \ln(K_{S_0}))/$$
$$(\ln(K_{S_d} - \ln(K_{S_0}))(K_{S_0} - K_{S_d})^5. \quad \text{(B2)}$$